

## **Degeneracy in Simplex Method**

A basic feasible solution of a simplex method is said to be degenerate basic feasible solution if at least one of the basic variable is zero and at any iteration of the simplex method more than one variable is eligible to leave the basis and hence the next simplex iteration produces a degenerate solution in which at least one basic variable is zero. This concept is known as tie.

A situation may arise at any iteration when two or more columns may have exactly the same  $C_j - Z_j$  value (+ve or -ve depending on the type of LPP).

In order to break this tie, the selection for key column (entering variable) can be made arbitrary,. However, the number of iterations required to arrive at the optimal solution can be minimized by adopting the following rules;

If there is a tie between two decision variables, then the selection can be made arbitrarily.

If there is a tie between decision variable and slack (or surplus) variable, then select the decision variable to enter into the basis first.

If there is a tie between two slack (or surplus) variables, then selection can be made arbitrarily.

Again, while solving LPP the situation may arise in which there is a tie between two or more basic variables for leaving the basis i.e minimum ratio to identify the basic variable to leave the basis is not unique or values of one or more basic variables in the solution values column ( $x_B$ ) become equal to zero. This causes the problem of degeneracy. However, if minimum ration is zero, then the iterations of simplex method are repeated (cycle) indefinitely without arriving at the optimal solution.

In most of the cases when there is a tie in the minimum ratios, the selection is made arbitrarily. However, the number of iterations required to arrive at the optimal solution can be minimized by applying the following rules;

Divide the coefficients of slack variables in the simplex table where degeneracy is detected by the corresponding positive numbers of the key column in the row, starting from left to right.

The row which contains smallest ratio comparing from left to right column- wise becomes the key row.

Remark: When there is a tie between a slack and artificial variables to leave the basis, the preference shall be given to artificial variable to leave the basis and there is no need to apply the procedure for resolving degeneracy under such cases.

**Example:** Solve the following LPP

$$\text{Max } Z = 3x_1 + 9x_2$$

subject to the constraints

$$x_1 + 4x_2 \leq 8$$

$$x_1 + 2x_2 \leq 4$$

and

$$x_1, x_2 \geq 0$$

**Solution:**

Adding slack variables  $S_1$  and  $S_2$  to the constraints, the problem can be expressed as;

$$\text{Max } Z = 3x_1 + 9x_2 + 0S_1 + 0S_2$$

subject to the constraints

$$x_1 + 4x_2 + S_1 = 8$$

$$x_1 + 2x_2 + S_2 = 4$$

and

$$x_1, x_2, S_1, S_2 \geq 0$$

The initial basic feasible solution is given in Table 1

Table 1: Initial Solution

		$C_j \rightarrow$	3	9	0	0	
$C_B$	<b>B</b>	$b(=x_B)$	$x_1$	$x_2$	$S_1$	$S_2$	Min.Ratio
0	$S_1$	8	1	4	1	0	$\frac{8}{4} = 2$
0	$S_2$	4	1	2	0	1	$\frac{4}{2} = 2$
$Z = 0$		$Z_j$	0	0	0	0	
		$C_j - Z_j$	3	9	0	0	
				↑			

From the Table 1,  $C_2 - Z_2$  is the largest positive value, therefore variable  $x_2$  is selected to enter into the basis. However, both variables  $S_1$  and  $S_2$ . This is an indication of the existence of degeneracy. To obtain the unique key row, apply the following procedure for resolving degeneracy.

Write coefficients of the slack variables as follows;

	Column	
Row	$S_1$	$S_2$
$S_1$	1	0
$S_2$	0	1

Dividing the coefficients by the corresponding element of the key column, we obtain the following ratios;

	Column	
Row	$S_1$	$S_2$
$S_1$	$1/4=1/4$	$0/4=0$
$S_2$	$0/2=0$	$1/2=1/2$

Comparing the ratios of the previous step from left to right column-wise, the minimum ratio occurs for the second row. Therefore, the variable  $S_2$  is selected to leave the basis. The new solution is obtain by performing the following row operations and

shown in Table 2

$$\begin{aligned}
 & R_2(new) \xrightarrow{\quad} \frac{R_2(old)}{2(\text{keyelement})} = (2, 1/2, 1, 0, 1/2) \\
 & R_1(new) \xrightarrow{\quad} R_1(old) - 4R_2(new) = (0, -1, 0, 1, -2)
 \end{aligned}$$

Table 2: Optimal Solution

		$C_j \xrightarrow{\quad}$	3	9	0	0
$C_B$	<b>B</b>	$b(=x_B)$	$x_1$	$x_2$	$S_1$	$S_2$
0	$S_1$	0	-1	0	1	-2
9	$x_2$	2	1/2	1	0	1/2
$Z = 18$		$Z_j$	9/2	9	0	9/2
		$C_j - Z_j$	-3/2	0	0	-9/2

Since all  $C_j - Z_j \leq 0$  in Table 2. Therefore, an optimal solution is arrived at  $x_1 = 0$ ,  $x_2 = 2$  and Max  $Z = 18$ .

## Types of Linear Programming Solution

### Alternative (Multiple) Optimal Solution

The alternative optimal solution can be obtained by considering the  $C_j - Z_j$  row of the simplex table. We know that an optimal solution to a maximization problem is reached if all  $C_j - Z_j \leq 0$ . What will happen if  $C_j - Z_j = 0$  for some non-basic variable columns in the optimal simplex table? Each entry in the  $C_j - Z_j$  indicates the contribution per unit of a particular variable in the objective function value if it is entered into the basis. Thus, if a non-basic variable corresponding to which  $C_j - Z_j = 0$  is entered into the basis, a new solution will be arrived at but the value of the objective function will not change.

**Example:** Solve the following LPP;

$$\text{Max } Z = 6x_1 + 4x_2$$

subject to the constraints

$$2x_1 + 3x_2 \leq 30$$

$$3x_1 + 2x_2 \leq 24$$

$$x_1 + x_2 \geq 3$$

and

$$x_1, x_2 \geq 0$$

**Solution:**

Adding slack variables  $S_1$ ,  $S_2$ , surplus variable  $S_3$  and artificial variable  $A_1$  in the constraint set the LPP becomes;

$$\text{Max } Z = 6x_1 + 4x_2 + 0S_1 + 0S_2 + S_3 - MA_1$$

subject to the constraints

$$2x_1 + 3x_2 + S_1 = 30$$

$$3x_1 + 2x_2 + S_2 = 24$$

$$x_1 + x_2 - S_3 + A_1 = 3$$

and

$$x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$$

The optimal solution for this LPP is presented in Table 1

Table 1: Optimal Solution

		$C_j \rightarrow$	6	4	0	0	0	
$C_B$	B	$b(=x_B)$	$x_1$	$x_2$	$S_1$	$S_2$	$S_3$	Min.Ratio
0	$S_1$	14	0	$5/3$	1	$-2/3$	0	$\frac{14}{5/3} = 42/5 \rightarrow$
0	$S_3$	5	0	$-1/3$	0	$1/3$	1	$\frac{5}{1/3} = 15$
6	$x_1$	8	1	$2/3$	0	$1/3$	0	$\frac{8}{2/3} = 12$
$Z = 48$		$Z_j$	6	4	0	2	0	
		$C_j - Z_j$	0	0	0	-2	0	
				↑				

The optimal solution shown in Table 1 is  $x_1 = 8, x_2 = 0$  and Max  $Z=48$ .

From the Table 1,  $C_2 - Z_2 = 0$  corresponding to a non-basic variable,  $x_2 = 0$ . Thus, an alternative optimal solution can also be obtained by entering variable  $x_2$  into the basis and removing  $S_1$  from the basis. The new solution is shown in Table 2

Table 2: Alternative Solution

		$C_j \rightarrow$	6	4	0	0	0
$C_B$	B	$b(=x_B)$	$x_1$	$x_2$	$S_1$	$S_2$	$S_3$
4	$x_2$	$42/5$	0	1	$3/5$	-	0
						$2/5$	
0	$S_3$	$39/5$	0	0	$1/5$	$1/5$	1
6	$x_1$	$12/5$	1	0	$-2/5$	$3/5$	0
$Z = 48$		$Z_j$	6	4	0	2	0
		$C_j - Z_j$	0	0	0	-2	0

The optimal solution shown in Table 2 is  $x_1 = 12/5, x_2 = 42/5$  and Max  $Z=48$ .

Further observe that in Table 4.24,  $C_3 - Z_3 = 0$  and variable  $S_1$  is not in the basis. This again indicates that an alternative optimal solution exists, thus for each alternative solution (infinite number of solutions) the value of objective function will remain the same.